Circle

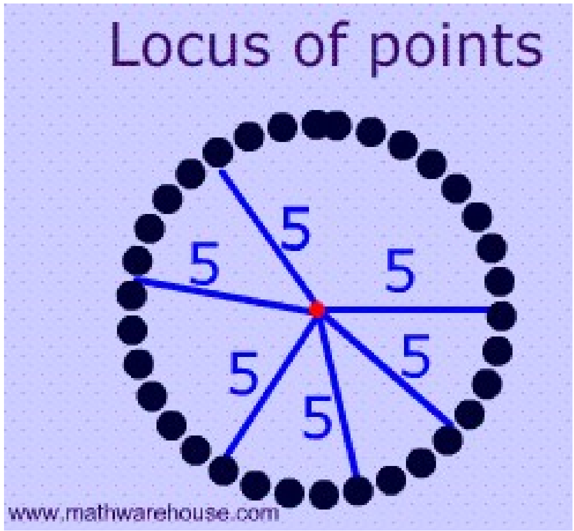
Revision of concepts

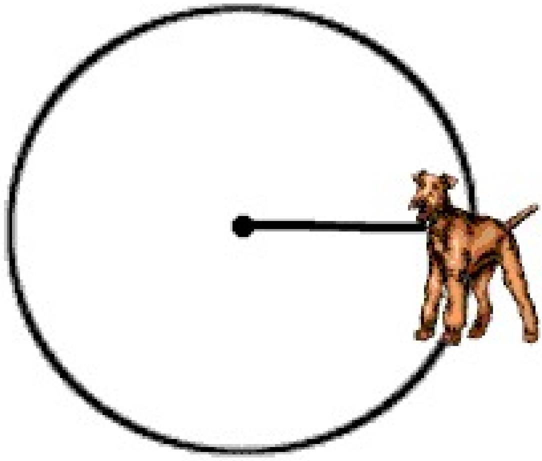
We will now revise:

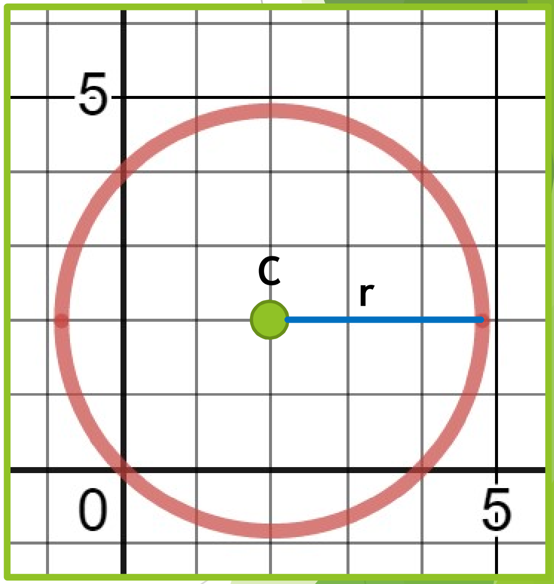
- Definition of a circle.
- Circles in nature
- Equations of a circle
  - Center – radius form
  - Diameter form
  - Parametric form
  - General equation
  - Circle touching the axes
  - Identifying a circle in the general equation of a conic section.
- Intercepts and Position of a point w.r.t. the circle
- Tangents
  - Condition for tangency
  - Slope form, point form, parametric form of tangents
- Normals
  - Point form
  - Slope form
- Using standard notations
  - Pair of tangents from external point
  - Length of tangents from external point
  - Chord of contact
  - Chord with a given mid point
- Director circle
- Angle of intersection of two circles and orthogonality
- Radical axis and its properties
- Length of common chord and common tangents.
- Family of circles.

Definition of a circle

- **Locus of a point which moves in a plane such that its distance from a fixed point in the plane is always a constant.**







- **For a given perimeter, a circle encloses the maximum area of any 2-D closed curve.**

Circles in nature

An underwater sinkhole

The human eye

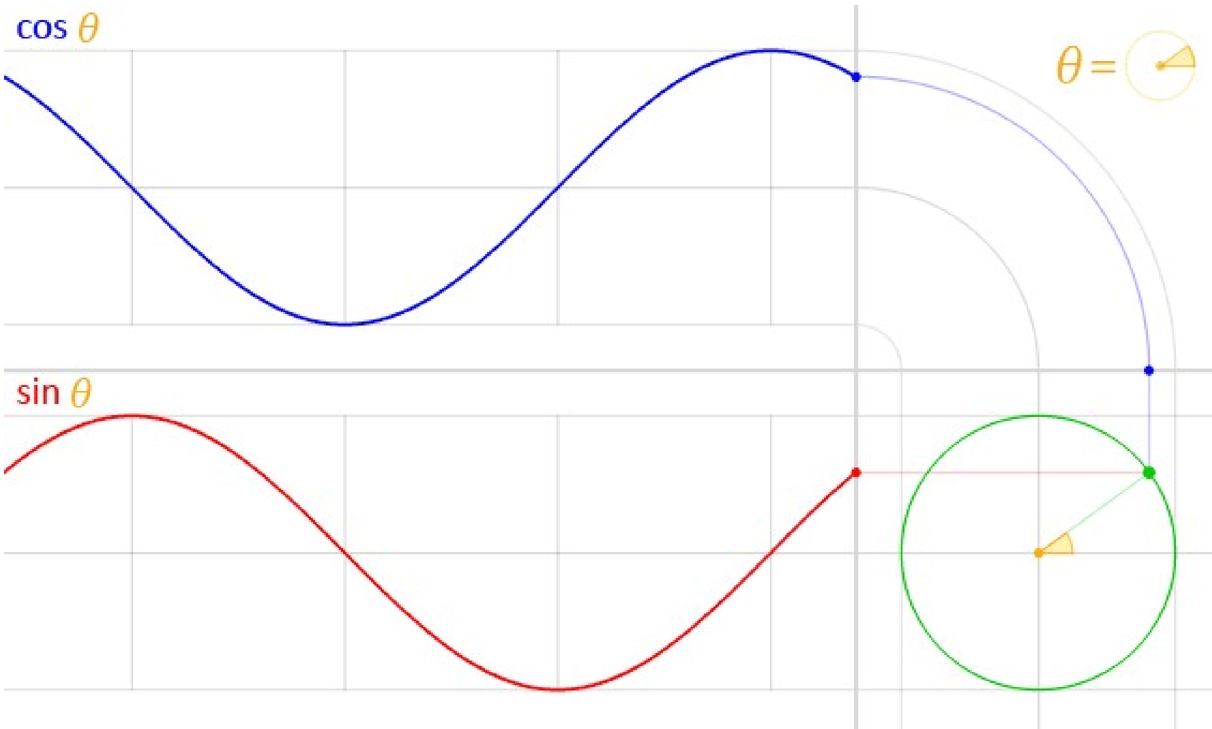
A full rainbow seen from an airplane

An underwater design by Japanese Puffer fish

Ripples in water

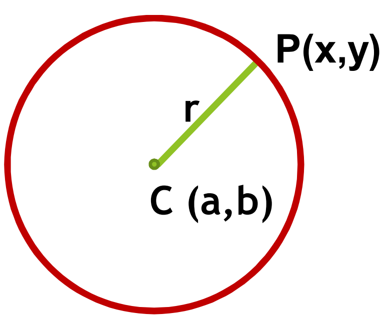
Growth rings in trees

Circles allow us to define sinθ and cosθ for all values of θ



Equations of a circle

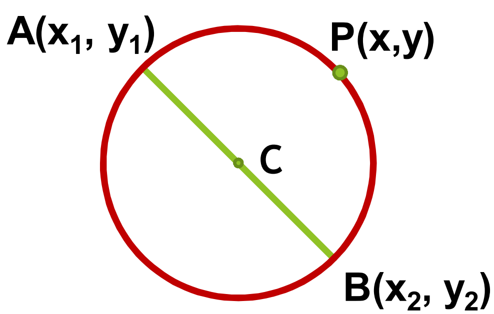
1. Centre - radius form



$(x - a)^2 + (y - b)^2 = r^2$

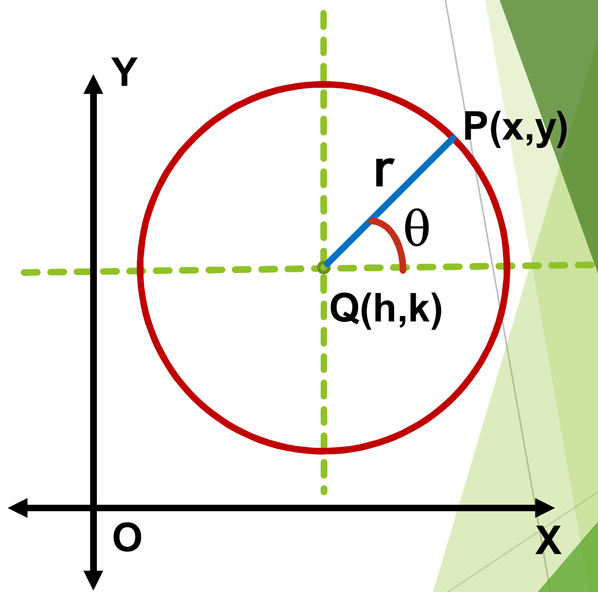
Note:
$(x - a)^2 + (y - b)^2 = 0$
is called a point circle. It represents C(a,b)

2. Diameter form



$(x – x_1)\ (x – x_2) +$
$(y – y_1)\ (y – y_2) = 0$

3. Parametric form



$x = h + r \cos \theta$
$y = k + r \sin \theta$

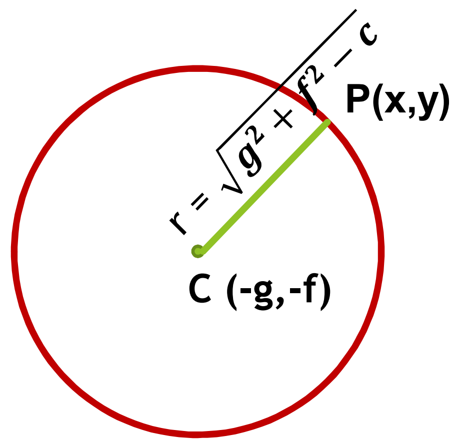
General equation of a circle

- **The general equation of a circle is**

$$x^2 + y^2 + 2gx + 2fy + c = 0 \ldots (1)$$

where g, f, and c are real constants.

- **Centre (- g, - f), radius, r =** $\sqrt{g^2 + f^2 - c}$



$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Note:

- A circle that touches X – axis will be of the form $x^2 + y^2 + 2gx + 2fy + g^2 = 0$
- A circle that touches Y – axis will be of the form $x^2 + y^2 + 2gx + 2fy + f^2 = 0$
- A circle that touches both axes will be of the form $x^2 + y^2 + 2gx + 2gy + g^2 = 0$
- When c = 0, the circle passes through the origin.

Identifying a circle from general equation of a conic section

- **A general equation of second degree**

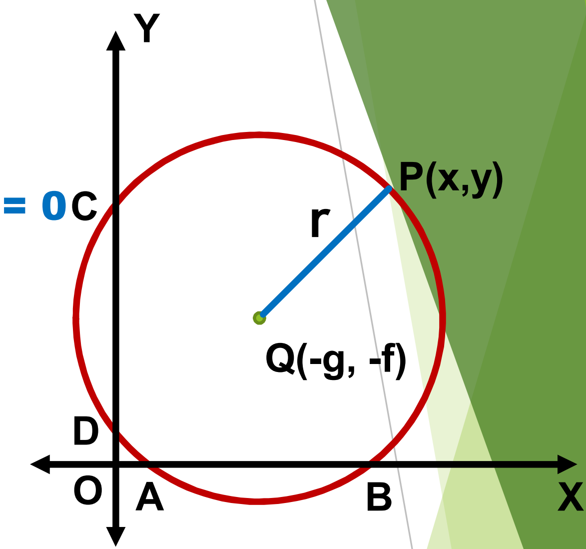
  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

  represents a circle if

- **$a = b$,**
- **$h = 0$ and**
- $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$
- **If $\Delta = 0$, it represents a point circle.**

Intercepts

- Intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$
  - on X-axis, is AB = $2\sqrt{g^2 - c}$
  - on Y-axis, is CD = $2\sqrt{f^2 - c}$
- When circle touches X – axis, $c = g^2$
- When circle touches Y – axis, $c = f^2$
- When circle touches both axes $c = g^2 = f^2$



Position of a point w.r.t. the circle

For a point $P(x_1,y_1)$, find the value of $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

If $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$, point is on the circle.

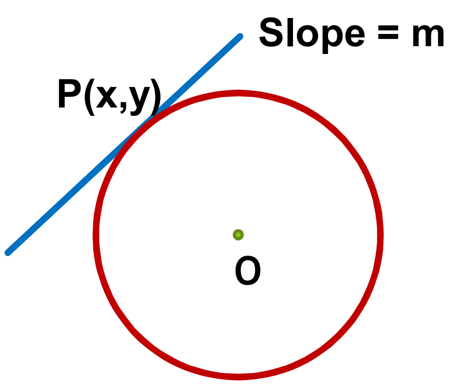
< 0, point is inside the circle.

> 0, point is outside the circle.

Tangents to standard circle $x^2 + y^2 = a^2$

Condition for tangency: $c = \pm a\sqrt{1+m^2}$ where $y = mx + c$ is the tangent.

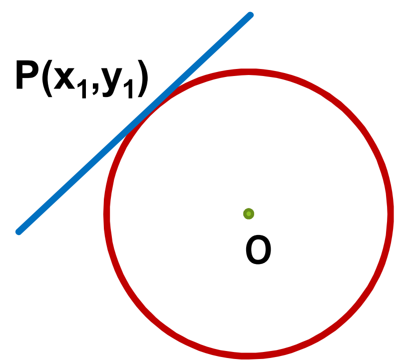
1. Slope form



$y = mx \pm a\sqrt{1+m^2}$

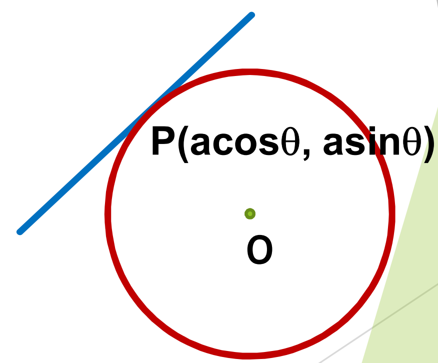
Note: If $P(\theta)$ is point of contact then $m = -\cot\theta$

2. Point form



$xx_1 + yy_1 = a^2$

3. Parametric form

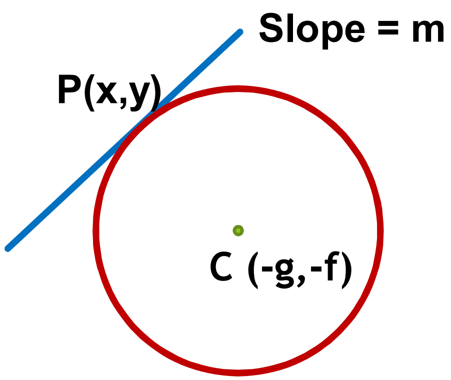


$x\cos\theta + y\sin\theta = a$

Tangents to a general circle
$x^2 + y^2 + 2gx + 2fy + c = 0$

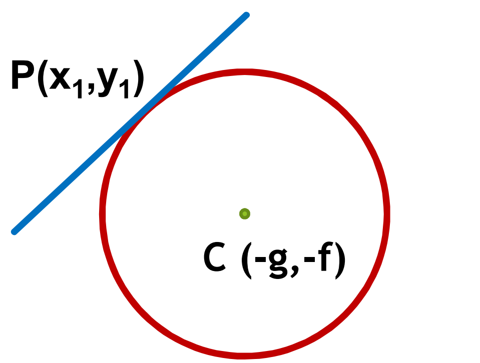
Condition for tangency: Apply the condition that distance (d) of center (-g, -f) from the given line = radius $\sqrt{g^2 + f^2 - C}$

1. Slope form



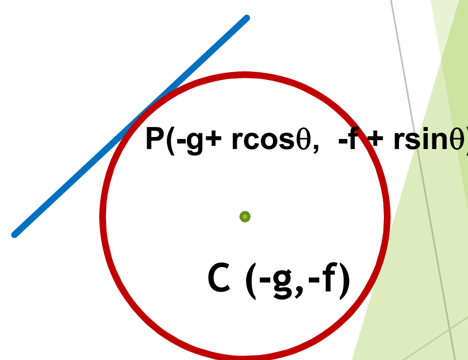
Take y = mx + c as tangent.
To find c, use d = r
where d = distance from center to line.

2. Point form



$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

3. Parametric form



Not commonly used for general circle.

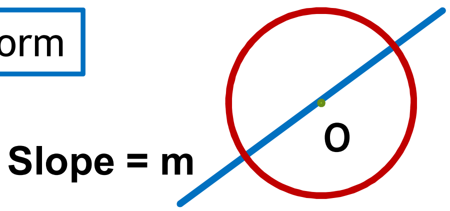
Normals

Condition for normalcy: Any line is normal if it passes through the center of the circle.

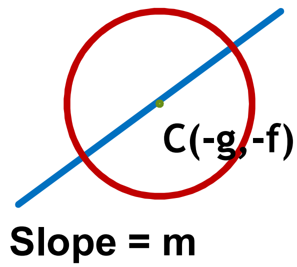
Standard circle
$x^2 + y^2 = a^2$

General circle
$x^2 + y^2 + 2gx + 2fy + c = 0$

Slope form



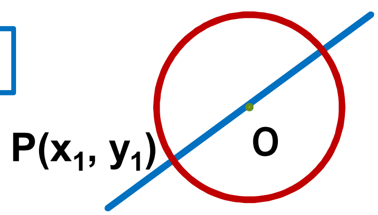
$$y = mx$$



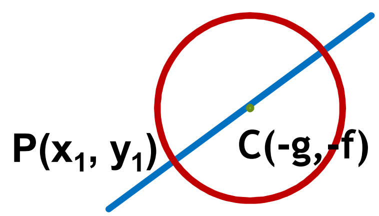
$$y = mx + c$$

Find c by substituting C(-g, -f) in y = mx + c

Point form



$$y = \frac{y_1}{x_1} x$$



$$\frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}$$

Standard Notations

- **Standard circle:**

  Let $S \equiv x^2 + y^2 - a^2,$

  $S_1 \equiv xx_1 + yy_1 - a^2,$

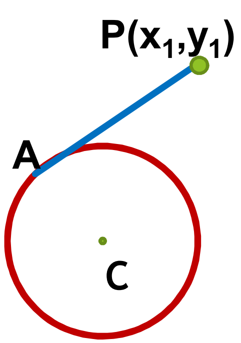
  $S_{11} \equiv x_1^2 + y_1^2 - a^2$

- **General Circle:**

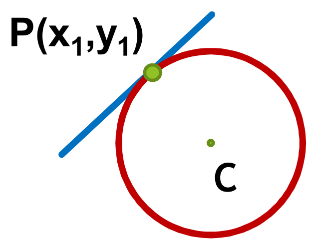
  Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0,$

  $S_1 \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

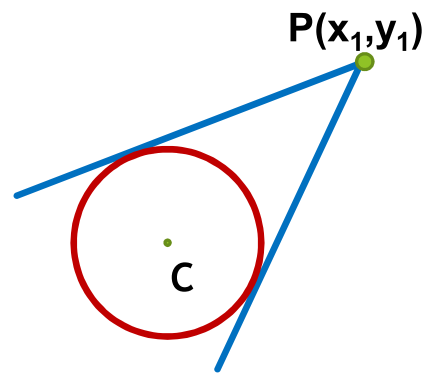
  $S_{11} \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$



Length of tangent PA from $P(x_1, y_1) = \sqrt{S_{11}}$



Equation of a tangent at $P(x_1, y_1)$ on the circle is $S_1 = 0$



Equation of a pair of tangents from $P(x_1, y_1)$ is $SS_{11} = S_1^2$

Standard Notations

- **Standard circle:**

  Let $S \equiv x^2 + y^2 - a^2$,

  $S_1 \equiv xx_1 + yy_1 - a^2$,

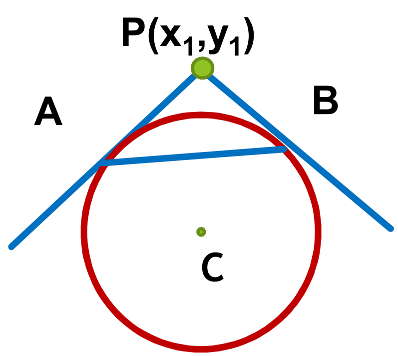
  $S_{11} \equiv x_1^2 + y_1^2 - a^2$

- **General Circle:**

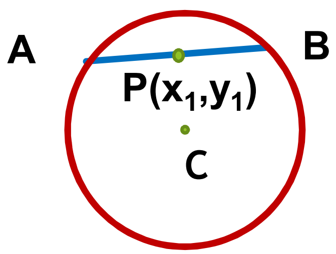
  Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$,

  $S_1 = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

  $S_{11} = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

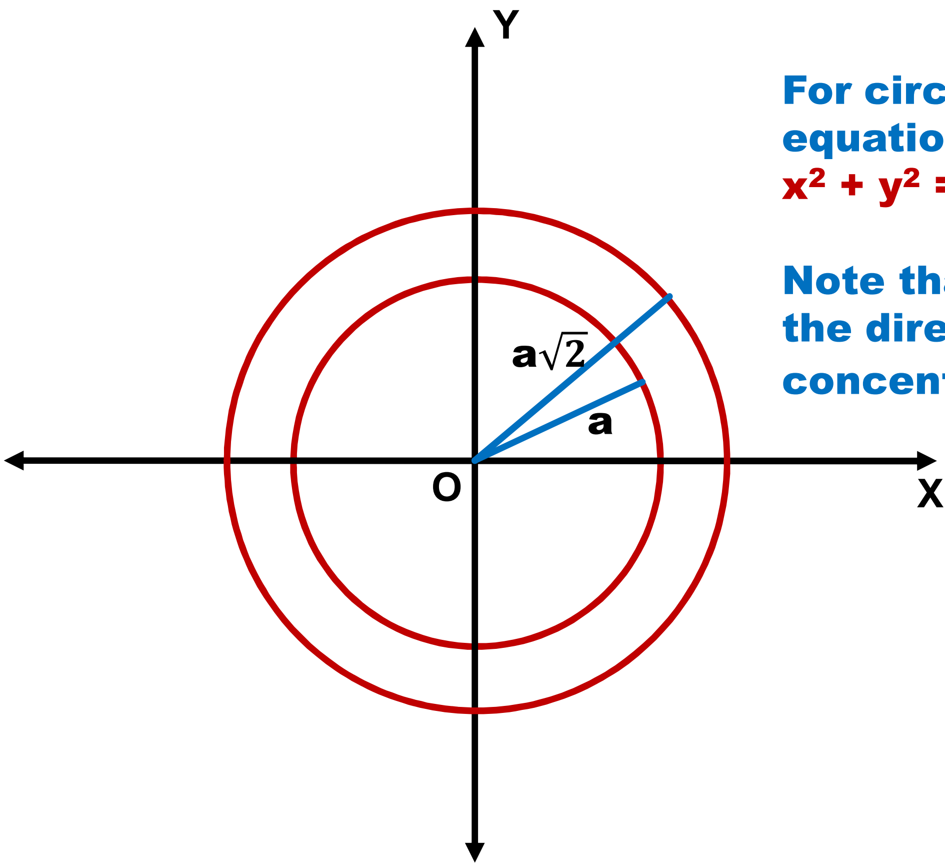


Equation of chord of contact AB of external point $P(x_1, y_1)$ is $S_1 = 0$



Equation of the chord AB bisected at $P(x_1, y_1)$ is $S_1 = S_{11}$.

Director Circle



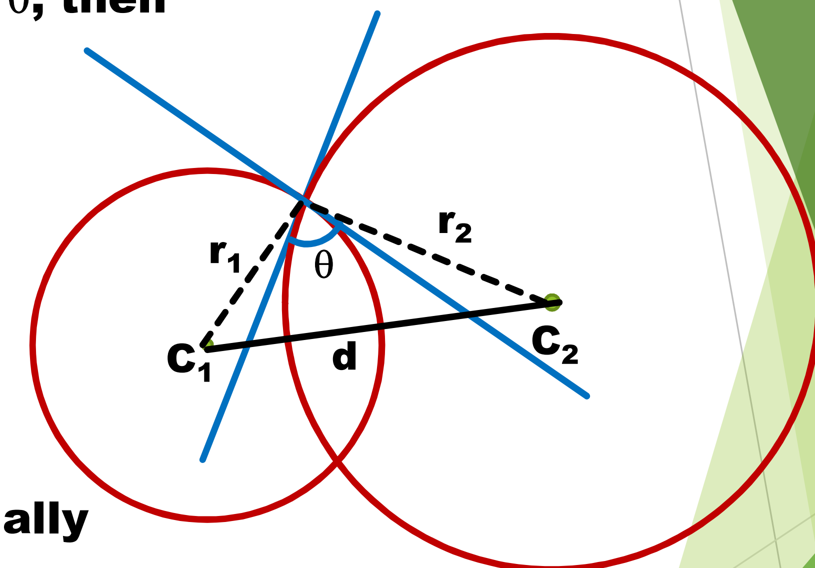
For circle $x^2 + y^2 = a^2$,
equation of director circle is
$x^2 + y^2 = 2a^2$

Note that if radius of circle is r,
the director circle is
concentric with radius $r\sqrt{2}$

Angle of intersection of circles

If two circles intersect at an angle θ, then

$$\cos\theta = \left|\frac{r_1^2+r_2^2-d^2}{2r_1r_2}\right|$$



Orthogonality

Two circles will intersect orthogonally i.e. θ = 90º, iff $2gg_1 + 2ff_1 = c + c_1$

where the circles are:

$x^2 + y^2 + 2gx + 2fy + c = 0$ and

$x_1^2 + y_1^2 + 2g_1x + 2f_1y + c_1 = 0$

Radical axis

If $S_1 = 0$ and $S_2 = 0$ are the equations of two circles, then radical axis is given by

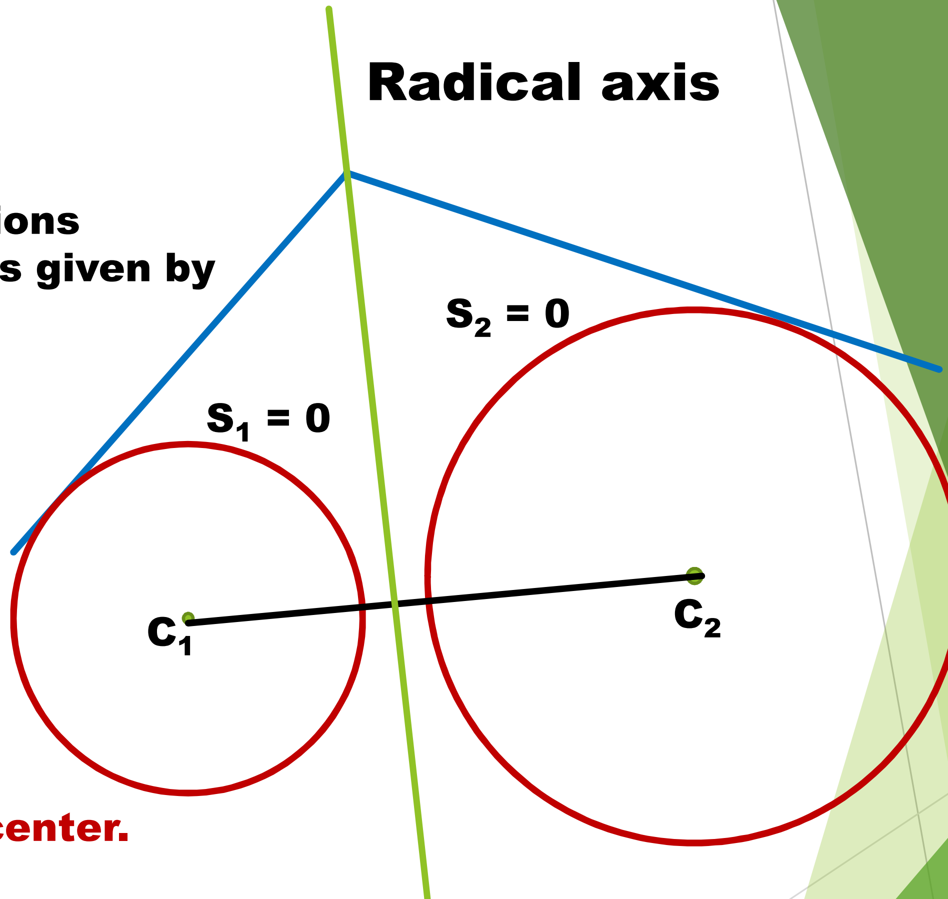
$$S_1 - S_2 = 0$$

Properties:

1. The radical axis **bisects** the common tangents of the two circles.

2. Radical axes of three circles taken pairwise are collinear. Common point is called **Radical center.**

3. When circles are intersecting, Radical axis is the **common chord** itself.

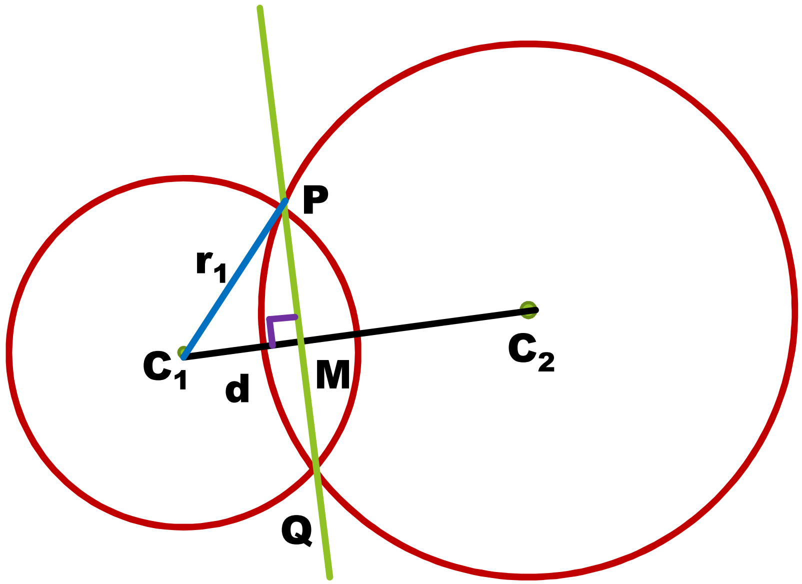


Common chord

PQ = 2PM

$= 2\sqrt{r_1^2 - d^2}$

where d is the distance of C_1 from the radical axis $S_1 - S_2 = 0$.

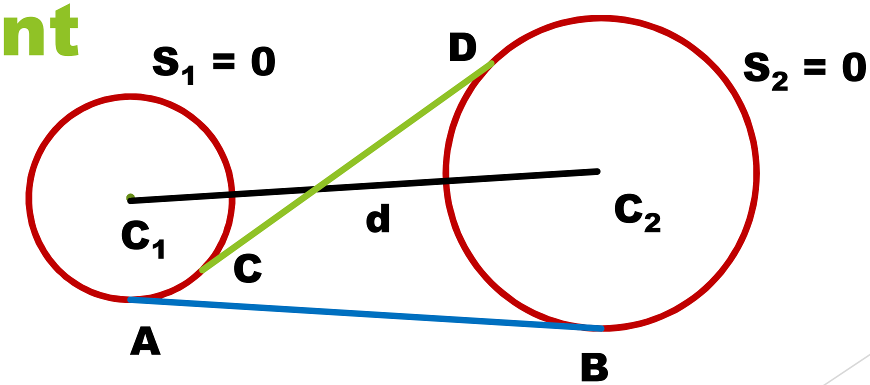


Common tangent segment

$AB = \sqrt{d^2 - (r_1 - r_2)^2}$

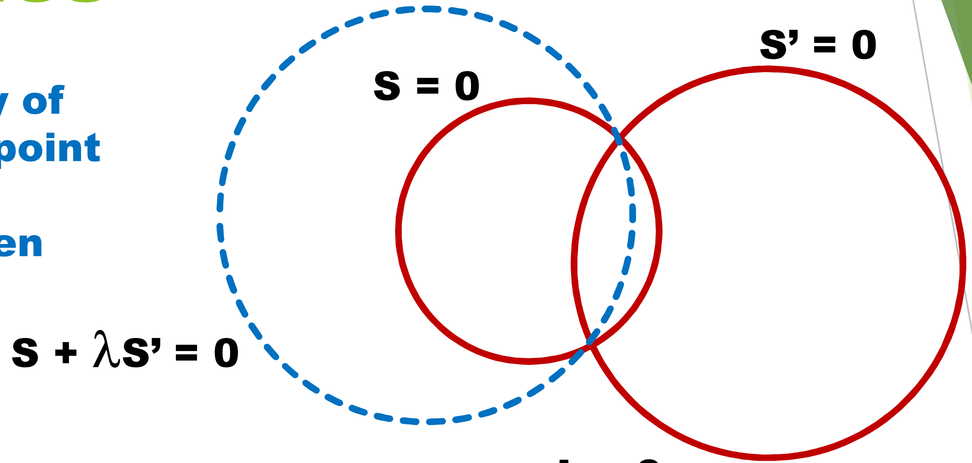
$CD = \sqrt{d^2 - (r_1 + r_2)^2}$

where $C_1C_2 = d$

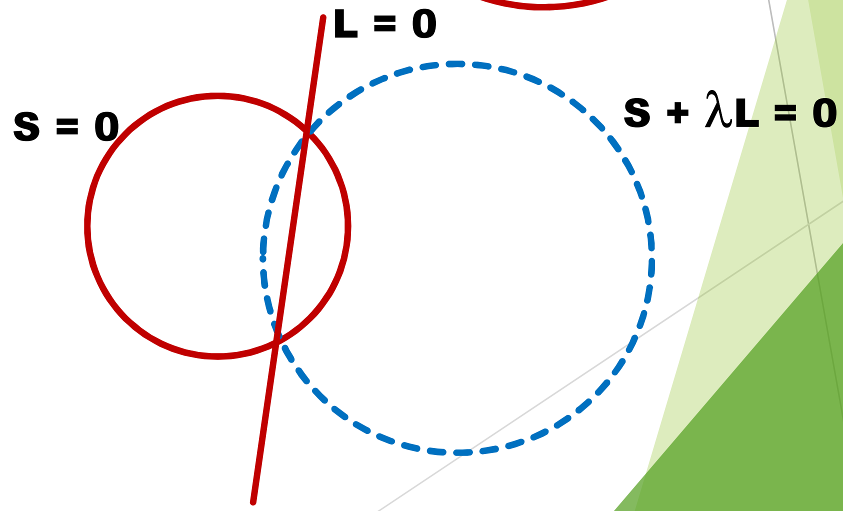


Family of circles

1. The equation of the family of circles passing through the point of intersection of two given circles S = 0 and S'= 0 is given as $S + \lambda S' = 0$ where λ is a parameter.



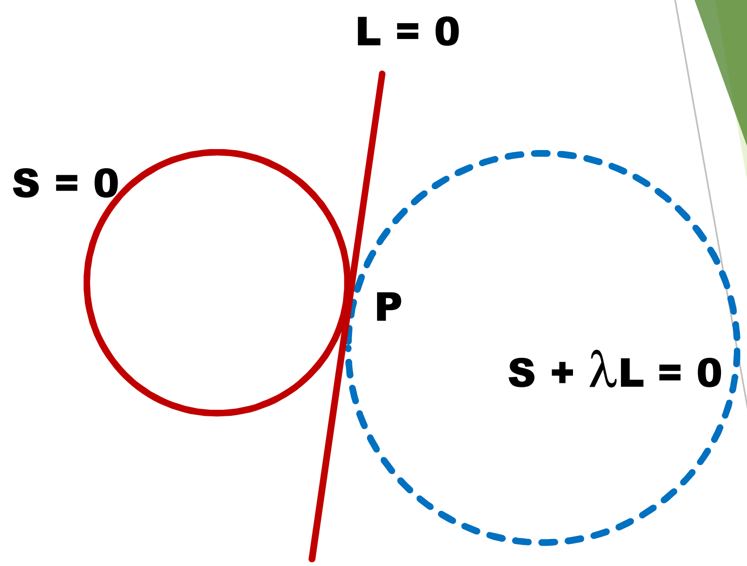
2. The equation of a family of circles passing through the line of intersection of a circle S = 0 and a line L = 0 is given as $S + \lambda L = 0$ where λ is a parameter.



Family of circles

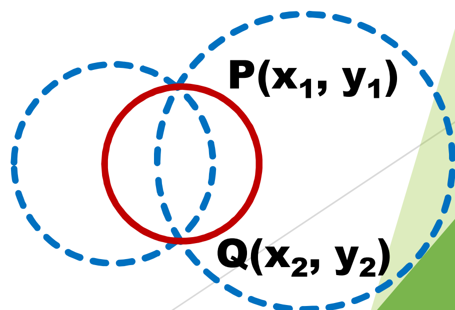
3. The equation of a family of circles touching the circle S = 0 and the line L = 0 at their point of contact P can be written as

$S + \lambda L = 0$



4. The equation of family of circles passing through two given points $P(x_1, y_1)$ and $Q\ (x_2, y_2)$ can be written in the form

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$



The End ☺